J. Fluid Mech. (1965), *vol.* 23, *part* 3, *pp.* 585-609
Prtnted in Great Britain

The theory for an oscillating thin airfoil as derived from the Oseen equations

By S. F. SHEN
Cornell University, Ithaca, New York
AND P. CRIMI
Cornell Aeronautical Laboratory Inc., Buffalo, New York

(Received 25 November 1964)

The classical potential solution for the flow about a thin airfoil in either steady or oscillatory motion requires application of the condition, postulated by Kutta, that the fluid velocity be finite at the trailing edge of the airfoil. The Kutta condition derives from the argument that viscous stresses will not allow a flow to turn about a sharp edge. Analytic verification of the validity of this condition, of particular interest in the unsteady case, has not previously been obtained. The problem is treated here by utilizing the Oseen formulation for viscous flow. The solution thus obtained approaches small-perturbation potential flow at a large distance from the airfoil and retains a qualitatively correct representation of the rotational flow near the airfoil. By simply assuming that the resultant force on the airfoil is finite, it is shown that the Kutta condition must apply in the limit of vanishing viscosity.

The first-order corrections, for large Reynolds number, to the lift and moment on an oscillating airfoil are explicitly determined. The effect of the Oseen approximation on the applicability of the numerical results remains to be established.

1. Introduction

The solution of the Navier–Stokes equations for large Reynolds number constitutes a singular perturbation of an inviscid flow, since in the limit of vanishing viscosity the derivatives of highest order are eliminated from the differential equations. This singular behaviour is particularly evident in the two-dimensional flow about a cylinder, where the potential solution admits an arbitrary value for the circulation about the cylinder. Classical airfoil theory is derived by removing the indeterminacy through application of the Kutta condition, which specifies that the rear stagnation point be located at the trailing edge. The use of this condition is justified by arguing that viscous effects must prevent the flow from turning around the sharp trailing edge. Experimentally determined flow patterns and loadings do confirm that the Kutta condition is valid, at least for steady flow. An analytic derivation of the Kutta condition has not been obtained up to now, however. That is, there has been no rigorous demonstration of the manner in which the potential flow derives from the solution to the complete Navier–Stokes equations in the limit as the viscosity vanishes.

Analytic determination of the Kutta condition is of more than just academic interest. Thin-airfoil theory has been notably successful in providing the aerodynamic loading on an oscillating airfoil for the prediction of flutter instabilities. The theory has been found to be inadequate, however, in predicting bending-torsion flutter of hydrofoils. Specifically, calculations indicate that no flutter should occur for fluid density greater than a certain critical value, while flutter has been obtained experimentally for fluid densities considerably greater than critical (see for example Henry 1962 and Woolston & Castile 1951). Since the structural properties of flutter models may be obtained to a high degree of accuracy, it is generally accepted that the representation of the unsteady hydrodynamic forces is responsible for the discrepancy. The validity of the Kutta condition for unsteady flow has therefore become open to question.

This investigation is directed to verifying analytically the validity of the Kutta condition and to determining the magnitude of direct viscous effects in the flow about an oscillating thin airfoil. It is claimed that a meaningful formulation of the problem may be obtained from the Oseen approximation to the Navier–Stokes equations. This approximation was originally devised for the flow over an arbitrary body at low Reynolds number (see for example Lamb 1945, p. 609).

Suppose that an airfoil is oscillating with small amplitude in a steady uniform stream. If it is then assumed that the disturbances caused by the airfoil are small throughout the flow and the Navier–Stokes equations are linearized accordingly, the resulting expressions are precisely the Oseen equations. Clearly, these equations must agree with the potential representation of thin-airfoil theory at some distance from the airfoil, where viscous effects are negligible. The correct momentum balance is also obtained from these equations at the surface of the airfoil, because the no-slip condition applies there.

These are the same arguments that motivated the use of the original Oseen approximation. There is no need here, though, to restrict application to low Reynolds number. That is, the only substantial error caused by the Oseen approximation comes from its simplified representation of the boundary layer, where streamwise convection is exaggerated. The basic mechanisms of momentum transfer have been retained. The Oseen approximation provides a suitable means for verifying the validity of the Kutta condition, then, since that condition derives from the limit of vanishing viscosity, and it is only required in taking this limit that viscous effects be correctly represented in the qualitative sense.

The Oseen approximation has been applied previously to the flow about a stationary flat plate in a uniform stream at large Reynolds number by Piercy & Winny (1933) and by Tamada & Miyagi (1962), the former for a plate aligned with the stream and the latter for the plate normal to the stream. Because of the symmetry, both papers necessarily dealt only with the drag problem. In the present application, consideration is given rather to the lifting problem, but the formulation used here, in terms of integral equations, directly parallels that used by these authors.

The analysis presented below proceeds from the consistent assumption of small perturbations in the equations of motion, as discussed above, and in the boundary conditions. The airfoil is then represented by an oscillating flat plate of infinite span. It is further assumed that the resultant force on the airfoil must be finite. The Kutta condition is then found to follow directly in the limit of vanishing viscosity.

The first-order corrections to lift and moment are explicitly determined, and are found to be $O(Re^{-\frac{1}{2}})$, Re denoting Reynolds number based on semichord, as would be expected from boundary-layer considerations. The applicability of these corrections is, of course, questionable, because the Oseen approximation provides a poor representation of the boundary layer. Their order of magnitude should be correct provided there is no extensive separation, however. Using these corrections, Crimi (1964) found only minor changes in the predicted flutter characteristics of a representative hydrofoil configuration. It appears, then, that the discrepancy between theory and experiment for flutter at high fluid density may not be attributed either to use of the Kutta condition or to direct viscous effects.

It should be mentioned that Chu (1962) also derived the viscous corrections to lift and moment on an oscillating flat plate. The analysis incorporates the Oseen approximation, but both the spirit and the method of approach differ from those applied here. Chu assumes that the Kutta condition holds, and in addition neglects viscous dissipation of the wake. He finds the viscous corrections to be $O(Re^{-1})$, in disagreement with the results below. The discrepancy appears to be mainly due to his use of an unsubstantiated assumption (see Crimi 1964).

2. Preliminary analysis

2.1. *Introduction of the Oseen approximation*

If U, b and b/U are chosen as units of speed, length and time, respectively, where U and b are characteristic to the flow, then the differential equations of plane, viscous incompressible flow may be written in dimensionless form as

$$\left.\begin{aligned} \frac{\partial q_x}{\partial x}+\frac{\partial q_y}{\partial y} &= 0, \\ \frac{Dq_x}{Dt}+\frac{\partial p}{\partial x} &= \frac{1}{Re}\nabla^2 q_x, \\ \frac{Dq_y}{Dt}+\frac{\partial p}{\partial y} &= \frac{1}{Re}\nabla^2 q_y, \end{aligned}\right\} \tag{2.1}$$

where Uq_x and Uq_y are components of fluid velocity in the directions of x and y, respectively, $\rho U^2 p$ is the static pressure, ρ being fluid density, and Re is the Reynolds number

$$Re = Ub/\nu$$

with ν denoting the kinematic viscosity. The operator D/Dt denotes the convective derivative with respect to time t and ∇^2 is the Laplacian operator.

The Oseen approximation is introduced as follows: let

$$q_x = 1+u,$$
$$q_y = v.$$

It is then assumed that quantities of second order in u and/or v are negligible in comparison with u, v, or their derivatives. This approximation is not valid near a boundary where the no-slip condition is imposed, u being of order unity in such a region. However, the correct differential relations are still satisfied at a boundary, due to the absence of convective terms there. Substituting for q_x and q_y in equations (2.1) and making this approximation, then, it is found that

$$\left.\begin{aligned} \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} &= 0, \\ \frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}+\frac{\partial p}{\partial x} &= \frac{1}{Re}\nabla^2 u, \\ \frac{\partial v}{\partial t}+\frac{\partial v}{\partial x}+\frac{\partial p}{\partial y} &= \frac{1}{Re}\nabla^2 v. \end{aligned}\right\} \tag{2.2}$$

The solution of equations (2.2) is implemented by assuming that u and v are each a sum of two contributions, one deriving from a potential $\phi(x,y,t)$ and the other rotational. Thus, let

$$\left.\begin{aligned} u &= u_p+u_r, \\ v &= v_p+v_r, \end{aligned}\right\} \tag{2.3}$$

where

$$\left.\begin{aligned} u_p &= \partial\phi/\partial x, \\ v_p &= \partial\phi/\partial y, \\ \nabla^2\phi &= 0. \end{aligned}\right\} \tag{2.4}$$

It then follows from equations (2.2) that the non-dimensional pressure p is given by

$$p = -\left(\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial t}\right), \tag{2.5}$$

and u_r and v_r are solutions of

$$\left.\begin{aligned} \nabla^2 u_r - 2k\left(\frac{\partial u_r}{\partial x}+\frac{\partial u_r}{\partial t}\right) &= 0, \\ \nabla^2 v_r - 2k\left(\frac{\partial v_r}{\partial x}+\frac{\partial v_r}{\partial t}\right) &= 0, \\ \frac{\partial u_r}{\partial x}+\frac{\partial v_r}{\partial y} &= 0, \end{aligned}\right\} \tag{2.6}$$

where $Re = 2k$.

2.2. *Formulation of the problem*

Consider a flat plate of infinite span and chord length two (or $2b$ in dimensional co-ordinates) immersed in a viscous, incompressible, uniform flow of unit speed (or speed U in dimensional variables). Assume that the plate is oscillating in pitch about its mid-chord point with amplitude α_0, and in a plunging mode with amplitude h_0, at a dimensionless frequency $\omega = \Omega b/U$, where Ω is the frequency of oscillation. The geometry may be represented as in figure 1.

The effect of the plate on the flow must be diminished with distance, requiring that

$$\left.\begin{aligned} \lim_{r\to\infty} u &= 0, \\ \lim_{r\to\infty} v &= 0, \end{aligned}\right\} \tag{2.7}$$

where

$$r = (x^2+y^2)^{\frac{1}{2}}.$$

Further, since the flow is viscous, the fluid velocity relative to the plate should vanish on the plate. As will be shown, use of the Oseen equations requires that the no-slip condition be applied on the interval $-1 \leqslant x \leqslant 1$ of the x-axis, rather than at $y = -h - x\alpha$. Specifically, the boundary conditions for small amplitudes of oscillation are

$$\left.\begin{aligned} u(x,0,t) &= -1 \\ v(x,0,t) &= -\frac{dh}{dt} - \alpha - x\frac{d\alpha}{dt} \end{aligned}\right\} \quad (-1 \leqslant x \leqslant 1). \qquad \begin{matrix}(2.8a)\\(2.8b)\end{matrix}$$

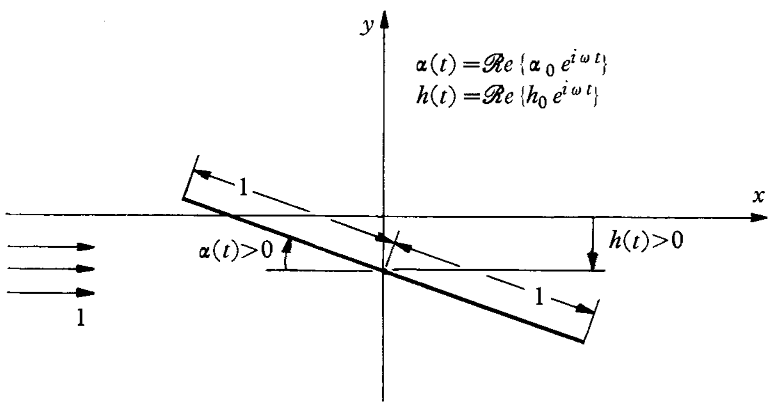

FIGURE 1. Representation of the geometry.

For harmonic motion, equation (2.8b) in complex notation becomes

$$v(x,0,t) = -[i\omega h_0 + (1 + i\omega x)\alpha_0]\, e^{i\omega t} \quad (-1 \leqslant x \leqslant 1). \qquad (2.8c)$$

The necessity for applying the no-slip condition at $y = 0$ rather than at the instantaneous position of the plate may be seen as follows. Consider the flow resulting from a plunging motion $h(t)$ (the argument is exactly analogous for pitching motion). Let u_e, v_e and p_e denote the solution of the exact non-linear equations, satisfying the exact boundary conditions, and let u_0, v_0 and p_0 denote the solution to the equations simplified by the Oseen approximation. We have, among other relations, that

$$\frac{Du_e}{Dt} \equiv \frac{\partial u_e}{\partial t} + (1 + u_e)\frac{\partial u_e}{\partial x} + v_e\frac{\partial u_e}{\partial y} = \frac{1}{Re}\nabla^2 u_e - \frac{\partial p_e}{\partial x}$$

and
$$u_e(x, -h, t) = -1 \quad (-1 \leqslant x \leqslant 1).$$

But specifying that u_e be constant on the plate is equivalent to

$$\left(\frac{Du_e}{Dt}\right)_{y=-h} = 0 \quad (-1 \leqslant x \leqslant 1).$$

It then follows that, on the plate, u_e must satisfy

$$\frac{1}{Re}\nabla^2 u_e - \frac{\partial p_e}{\partial x} = 0.$$

Next, consider the solution under the Oseen approximation. This solution satisfies

$$\frac{\partial u_0}{\partial t}+\frac{\partial u_0}{\partial x}=\frac{1}{Re}\nabla^2 u_0-\frac{\partial p_0}{\partial x}.$$

Now, if the plate is regarded as in its mean position at $y=0$, u_0 is restricted by equation (2.8a). Then, of course,

$$\left(\frac{\partial u_0}{\partial x}\right)_{y=0}=\left(\frac{\partial u_0}{\partial t}\right)_{y=0}\equiv 0 \quad (-1\leqslant x\leqslant 1),$$

and it follows immediately that

$$\frac{1}{Re}\nabla^2 u_0-\frac{\partial p_0}{\partial x}=0$$

at the mean position of the plate. This agrees with the momentum relation satisfied by the exact solution. If, on the other hand, the boundary conditions were applied at $y=-h$, i.e. if

$$\left.\begin{aligned} u_0(x,-h,t)&=-1\\ v_0(x,-h,t)&=-\mathring{h}\end{aligned}\right\} \quad (-1\leqslant x\leqslant 1),$$

then
$$\left(\frac{Du_0}{Dt}\right)_{y=-h}=0=\left[\frac{\partial u_0}{\partial t}+(1+u_0)\frac{\partial u_0}{\partial x}+v_0\frac{\partial u_0}{\partial y}\right]_{y=-h},$$

whence
$$0=\left(\frac{\partial u_0}{\partial t}-\mathring{h}\frac{\partial u_0}{\partial y}\right)_{y=-h}.$$

Thus,
$$\left(\frac{\partial u_0}{\partial t}+\frac{\partial u_0}{\partial x}\right)_{y=-h}=\mathring{h}\left(\frac{\partial u_0}{\partial y}\right)_{y=-h}\neq 0,$$

and u_0 would have to satisfy

$$\frac{1}{Re}\nabla^2 u_0-\frac{\partial p_0}{\partial x}=\mathring{h}\frac{\partial u_0}{\partial y}$$

on the plate, so the correct differential relation would not be reproduced at the plate. In an exactly similar manner, it is found that the correct differential relation for v_0 at the plate is only obtained by satisfying the boundary conditions at $y=0$. In other words, application of the Oseen approximation amounts to considering all convective changes to be in the direction of x at the free-stream speed. Any convective changes resulting from the transverse motion of the plate must then be represented by changes in time at a fixed position.

It should be noted, in regard to the above discussion, that apparently the term $\mathring{h}\,\partial u_0/\partial y$ is of higher order and may consistently be discarded from the expansion of u_0 about $y=0$; the boundary condition could then be applied at $y=0$ without resorting to such elaborate arguments. This term is not necessarily of higher order, however. For, if the Reynolds number is large, the term would be negligibly small in comparison with u_0 only if the amplitude of the motion is much less than the boundary-layer thickness. On the other hand, by satisfying the correct differential relations at the mean position of the plate, the no-slip boundary condition may be imposed there without restricting the amplitude of the motion relative to the boundary-layer thickness.

In connexion with the boundary conditions, it will be necessary to provide a representation of the wake emanating from the trailing edge of the plate. By the usual arguments, the displacement of the wake from the x-axis can be neglected. However, the viscous dissipation of the wake should be retained. Hence, the wake will be regarded as a sheet of decaying vortices, each element of which is convected downstream at the free-stream speed along the x-axis.

3. Derivation of the integral equations

3.1. *Representation of the plate*

Let subscripts (1) and (2) denote contributions to u and v of plate and wake singularity distributions, respectively. It may be verified that the following are solutions of equations (2.2):

$$\left.\begin{aligned} u_1(x,y,t) &= \frac{1}{2\pi}\int_{-1}^{1}\sigma(\xi)\left[\left\{\frac{1}{r}-k\,e^{k(x-\xi)}K_1(kr)\right\}\cos\theta-k\,e^{k(x-\xi)}K_0(kr)\right]d\xi \\ &\quad+\frac{e^{i\omega t}}{2\pi}\int_{-1}^{1}\gamma(\xi)\left[\frac{1}{r}-\beta\,e^{k(x-\xi)}K_1(\beta r)\right]\sin\theta\,d\xi, \\ v_1(x,y,t) &= \frac{1}{2\pi}\int_{-1}^{1}\sigma(\xi)\left[\frac{1}{r}-k\,e^{k(x-\xi)}K_1(kr)\right]\sin\theta\,d\xi \\ &\quad-\frac{e^{i\omega t}}{2\pi}\int_{-1}^{1}\gamma(\xi)\left[\left\{\frac{1}{r}-\beta\,e^{k(x-\xi)}K_1(\beta r)\right\}\cos\theta+k\,e^{k(x-\xi)}K_0(\beta r)\right]d\xi. \end{aligned}\right\} \tag{3.1}$$

The variables r and θ are polar co-ordinates with origin at $x=\xi$, $y=0$; the angle θ increases in a counter-clockwise direction and is zero for $y=0$, $x>\xi$. The functions K_0 and K_1 are Bessel functions of the second kind for imaginary argument, of order zero and one, respectively; β is the complex constant defined by

$$\beta^2 = k^2+2i\omega k,$$

where β is the square root having positive real part. K_0 and K_1 are defined for complex argument as follows (see Gray, Mathews & MacRobert 1952):

$$\left.\begin{aligned} K_0(z) &= \int_1^\infty \frac{e^{-z\lambda}\,d\lambda}{(\lambda^2-1)^{\frac{1}{2}}} \\ K_1(z) &= \int_1^\infty \frac{\lambda\,e^{-z\lambda}\,d\lambda}{(\lambda^2-1)^{\frac{1}{2}}} \end{aligned}\right\} \quad (\mathscr{R}\{z\}>0).$$

The terms in equations (3.1) containing Bessel functions are solutions of equations (2.6). The functions multiplying $\sigma(\xi)$ may be identified as representing a source-like flow, while those functions multiplying $\gamma(\xi)$ correspond to a vortical flow. Equations (3.1) generalize the solution for steady flow given by Tamada & Miyagi (1962). It may be verified that these expressions for u_1 and v_1 will be finite throughout the plane, provided $\sigma(x)$ and $\gamma(x)$ are themselves integrable. The latter requirement only implies that the forces on the plate must be finite.

3.2. *Representation of the wake*

The elemental solution for the wake must, when viewed from stream-fixed co-ordinates, be that of a fixed, isolated vortex which decays in a manner prescribed by the Oseen approximation. The solution sought may be found in Lamb

(1945, p. 592). Transforming to a plate-fixed co-ordinate system and forming a distribution of these singularities of semi-infinite extent, the induced velocities due to the wake are given by

$$\left.\begin{aligned} u_2(x,y,t) &= \frac{1}{2\pi}\int_1^\infty \epsilon(t+1-\xi)\left[1-\exp\left(-\frac{kr^2}{2(\xi-1)}\right)\right]\frac{\sin\theta}{r}\,d\xi, \\ v_2(x,y,t) &= -\frac{1}{2\pi}\int_1^\infty \epsilon(t+1-\xi)\left[1-\exp\left(-\frac{kr^2}{2(\xi-1)}\right)\right]\frac{\cos\theta}{r}\,d\xi, \end{aligned}\right\} \quad (3.2)$$

where r and θ are as defined for equations (3.1). The integrands in equations (3.2) may be interpreted physically as the induced velocities due to a vortex, with strength upon release of $\epsilon(t)$, located at $x = \xi$, $y = 0$.

In order to completely define the wake, it is necessary to obtain a relation between the strengths of the bound and shed vortices. The existence of such a relation can be argued on physical grounds. As the lift on the plate changes with time, the total circulation about the plate must change. But instantaneously the total circulation in the flow must be constant, since viscous dissipation requires a finite time interval to act. The total circulation is maintained by the shedding of a vortex into the wake whose strength is equal in magnitude and opposite in sign to the change in circulation about the plate.

This relation may be obtained as follows. If circulation is defined to be positive in the sense of positive lift (i.e. clockwise), then the circulation Γ_C about a closed circuit C is defined by

$$\Gamma_C = \oint_C \mathbf{q}\,.\,d\mathbf{l},$$

where $\mathbf{q}$ is the fluid velocity and $d\mathbf{l}$ is the vector of differential-length directed tangent to C in the direction of integration. If it is then assumed that C is being convected with the fluid and that quadratic terms can be neglected, consistent with the Oseen approximation, it is found that the time rate of change of Γ_C, necessarily convective, is given by

$$\frac{D\Gamma_C}{Dt} = \oint_C \left(\frac{\partial \mathbf{q}_r}{\partial x} + \frac{\partial \mathbf{q}_r}{\partial t}\right).\,d\mathbf{l}, \quad (3.3)$$

where $\mathbf{q}_r$ is the rotational contribution to $\mathbf{q}$. The details of the derivation of equation (3.3) are given by Crimi (1964).

In order to apply equation (3.3) to the problem being treated here, consider the circuit C shown in figure 2. Suppose that C is allowed to convect with the fluid. The circulation Γ_C about C may be computed, using the expressions for $u = u_1 + u_2$ and $v = v_1 + v_2$ as given above. It is then found that in the limit $R \to \infty$ (see Crimi 1964)

$$\frac{D\Gamma_C}{Dt} = \frac{d\Gamma_p}{dt} + \epsilon(t) - \left[\frac{\partial \mathscr{L}(x,t)}{\partial x} + \frac{\partial \mathscr{L}(x,t)}{\partial t}\right]_{x=1}, \quad (3.4)$$

where

$$\Gamma_p = e^{i\omega t}\int_{-1}^{1} \gamma(x)\,dx,$$

$$\mathscr{L}(x,t) = \int_{-\infty}^{\infty} v_r(x,y,t)\,dy,$$

with v_r being the sum of the rotational contributions to v_1 and v_2 as specified by equations (3.1) and (3.2). But upon substituting for $\mathbf{q}_r$ in equation (3.3), with C as defined in figure 2, it follows that

$$\frac{D\Gamma_C}{Dt} = -\left[\frac{\partial \mathscr{L}(x,t)}{\partial x} + \frac{\partial \mathscr{L}(x,t)}{\partial t}\right]_{x=1} \tag{3.5}$$

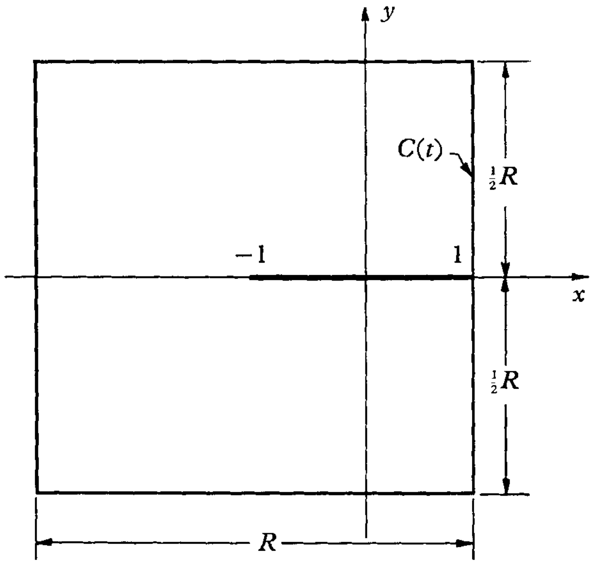

FIGURE 2. The circuit for relating the strengths of the bound and shed vortices.

Equating the results of equations (3.4) and (3.5), the integrals on v_r cancel to give

$$\epsilon(t) = -i\omega\overline{\Gamma}\, e^{i\omega t}, \tag{3.6}$$

where

$$\overline{\Gamma} = \int_{-1}^{1} \gamma(x)\, dx.$$

3.3. *The integral equations for σ and γ*

With the strengths of the wake singularities prescribed in terms of those of the plate according to equation (3.6), the flow field is completely defined by the unknown source and vortex strengths $\sigma(x)$ and $\gamma(x)$. Specifically, $u = u_1 + u_2$ and $v = v_1 + v_2$ are given by

$$\begin{aligned} u(x,y,t) = {} & \frac{1}{2\pi}\int_{-1}^{1} \sigma(\xi)\left[\left\{\frac{1}{r} - k\, e^{k(x-\xi)} K_1(kr)\right\}\cos\theta - k\, e^{k(x-\xi)} K_0(kr)\right] d\xi \\ & + \frac{e^{i\omega t}}{2\pi}\int_{-1}^{1} \gamma(\xi)\left[\frac{1}{r} - \beta\, e^{k(x-\xi)} K_1(\beta r)\right]\sin\theta\, d\xi \\ & - \frac{i\omega\overline{\Gamma}\, e^{i\omega t}}{2\pi}\int_{1}^{\infty} \exp\left[-i\omega(\xi-1)\right]\left[1 - \exp\left(-\frac{kr^2}{2(\xi-1)}\right)\right]\frac{\sin\theta}{r}\, d\xi, \end{aligned} \tag{3.7}$$

$$\begin{aligned} v(x,y,t) = {} & \frac{1}{2\pi}\int_{-1}^{1} \sigma(\xi)\left[\frac{1}{r} - k\, e^{k(x-\xi)} K_1(kr)\right]\sin\theta\, d\xi \\ & - \frac{e^{i\omega t}}{2\pi}\int_{-1}^{1} \gamma(\xi)\left[\left\{\frac{1}{r} - \beta\, e^{k(x-\xi)} K_1(\beta r)\right\}\cos\theta + k\, e^{k(x-\xi)} K_0(\beta r)\right] d\xi \\ & + \frac{i\omega\overline{\Gamma}\, e^{i\omega t}}{2\pi}\int_{1}^{\infty} \exp\left[-i\omega(\xi-1)\right]\left[1 - \exp\left(-\frac{kr^2}{2(\xi-1)}\right)\right]\frac{\cos\theta}{r}\, d\xi. \end{aligned} \tag{3.8}$$

By setting y equal to zero in equation (3.7) and substituting the resulting expression for $u(x,0,t)$ in equation (2.8a), the following integral equation for $\sigma(x)$ is obtained:

$$-1 = \frac{1}{2\pi}\int_{-1}^{1}\sigma(\xi)\left\{\frac{1}{x-\xi} - k\,\mathrm{sgn}\,(x-\xi)\,e^{k(x-\xi)}K_1(k|x-\xi|)\right.$$
$$\left. - k\,e^{k(x-\xi)}K_0(k|x-\xi|)\right\}d\xi \quad (-1 \leqslant x \leqslant 1); \qquad (3.9)$$

where
$$\mathrm{sgn}\,(x-\xi) = 1 \qquad (x > \xi);$$
$$\mathrm{sgn}\,(x-\xi) = -1 \quad (x \leqslant \xi).$$

The approximate solutions to equation (3.9), both for $k \ll 1$ and for $k \gg 1$, are derived by Piercy & Winny (1933). Since $\sigma(x)$ contributes only to the drag, the solution of equation (3.9) is not of immediate interest here and so will not be discussed.

The integral equation for $\gamma(x)$ is obtained by setting y to zero in equation (3.8) and substituting for $v(x,0,t)$ in equation (2.8c). The result is

$$(1+i\omega t)\,\alpha_0 + i\omega h_0 = \frac{1}{2\pi}\int_{-1}^{1}\gamma(\xi)\left\{\frac{1}{x-\xi} - \beta\,\mathrm{sgn}\,(x-\xi)\,e^{k(x-\xi)}K_1(\beta|x-\xi|)\right.$$
$$\left. + k\,e^{k(x-\xi)}K_0(\beta|x-\xi|)\right\}d\xi$$
$$+\frac{i\omega\Gamma}{2\pi}\int_{1}^{\infty}\exp\left[-i\omega(\xi-1)\right]\left[1-\exp\left(-\frac{k(\xi-x)^2}{2(\xi-1)}\right)\right]\frac{d\xi}{\xi-x} \quad (-1 \leqslant x \leqslant 1). \qquad (3.10)$$

The solution of equation (3.10) provides the pressure, and hence the lift and moment, exerted on the plate. Specifically, from equation (2.5), it follows that

$$\Delta p \equiv p(x,0^-,t) - p(x,0^+,t)$$
$$= \left\{\gamma(x) + i\omega\int_{-1}^{x}\gamma(\xi)\,d\xi\right\}e^{i\omega t}. \qquad (3.11)$$

Upon integration, the following expressions for lift L per unit span and moment M per unit span about mid-chord (positive to increase angle of attack) are obtained:

$$\frac{L(t)}{\rho U^2 b} = \left[\Gamma + i\omega\int_{-1}^{1}(1-x)\,\gamma(x)\,dx\right]e^{i\omega t}, \qquad (3.12)$$

$$\frac{M(t)}{\rho U^2 b^2} = \left[-\int_{-1}^{1}x\gamma(x)\,dx - \tfrac{1}{2}i\omega\int_{-1}^{1}(1-x^2)\,\gamma(x)\,dx\right]e^{i\omega t}. \qquad (3.13)$$

4. Analysis of the lifting problem—the Kutta condition

In the following analysis, the functional form of $\gamma(x)$ is first deduced. Then the orders of magnitude, for large Reynolds number, of the contributions of the various terms of $\gamma(x)$ to the integral equation are obtained. It is then possible to determine the form of the solution in the limit of infinite Reynolds number and to compute a first correction to the inviscid solution for large but finite Reynolds number.

4.1. *Determination of the functional form of $\gamma(x)$*

If both sides of equation (3.10) are differentiated with respect to x, the result is to change the character of the singular part of the integrand from logarithmic to algebraic. The singularity obtained by differentiation is of the form $(x-\xi)^{-1}$, the integral being defined as the Cauchy principal value. Performing the differentiation, then, and subtracting out the singular part of the integrand, it is found that

$$\frac{1}{2\pi}\mathscr{P}\int_{-1}^{1}\frac{\gamma(\xi)\,d\xi}{x-\xi}=f_1(x)+f_2(x)\quad(-1\leqslant x\leqslant 1),\tag{4.1}$$

where

$$f_1(x)=\frac{1}{2\pi k}\int_{-1}^{1}\gamma(\xi)\left[\frac{-1}{(x-\xi)^2}+\frac{k}{x-\xi}+e^{k(x-\xi)}\{(k^2+\beta^2)\,K_0(\beta|x-\xi|)\right.$$
$$\left.-2\beta k\operatorname{sgn}(x-\xi)\,K_1(\beta|x-\xi|)+\tfrac{1}{2}\beta^2K_2(\beta|x-\xi|)\}\right]d\xi,$$

and

$$f_2(x)=-\frac{i\omega\alpha_0}{k}+\frac{i\omega\overline{\Gamma}}{2\pi k}\frac{d}{dx}\int_{1}^{\infty}\exp[-i\omega(\xi-1)]\left[1-\exp\left(-\frac{k(\xi-x)^2}{2(\xi-1)}\right)\right]\frac{d\xi}{\xi-x}.$$

The $\mathscr{P}$ preceding the integral signifies that the Cauchy principal value is to be taken.

It should be observed that if $\gamma(x)$ is assumed to have integrable singularities (i.e. that lift is finite), then $f_1(x)$ is bounded over the whole interval $-1\leqslant x\leqslant 1$. Further, provided that $x<1$, it is permissible to interchange the order of integration and differentiation in the expression for $f_2(x)$. The resulting integral is bounded so, for $x<1, f_2(x)$ is also bounded.

However, $f_2(x)$ has a singularity at $x=1$. The nature of the singularity may be determined by deriving an equivalent expression for the integral in question, whereupon the behaviour at $x=1$ becomes evident. The details of the calculation are given by Crimi (1964). The result is that

$$\lim_{x\to 1}\left[\frac{1}{\ln(1-x)}f_2(x)\right]=\frac{i\omega\overline{\Gamma}}{2\pi}.\tag{4.2}$$

The general form of the solution to equation (3.10) may now be deduced. To do this, let

$$f_0(x)=f_1(x)+f_2(x)-(i\omega\overline{\Gamma}/2\pi)\ln(1-x).$$

Thus, equation (4.1) may be written

$$\frac{1}{2\pi}\mathscr{P}\int_{-1}^{1}\frac{\gamma(\xi)\,d\xi}{x-\xi}=f_0(x)+\frac{i\omega\overline{\Gamma}}{2\pi}\ln(1-x)\quad(-1\leqslant x\leqslant 1),\tag{4.3}$$

where now $f_0(x)$ is bounded over the whole interval $-1\leqslant x\leqslant 1$. But

$$\frac{1}{2\pi}\mathscr{P}\int_{-1}^{1}[-\tfrac{1}{2}i\omega\overline{\Gamma}(1+\xi)]\frac{d\xi}{x-\xi}=\frac{i\omega\overline{\Gamma}}{2\pi}\left\{\left(\frac{1+x}{2}\right)\ln\left(\frac{1-x}{1+x}\right)-1\right\}.$$

Therefore, let $$\gamma(x)=\hat{\gamma}(x)-\tfrac{1}{2}i\omega\overline{\Gamma}(1+x),$$

so that $\hat{\gamma}$ must satisfy

$$\frac{1}{2\pi}\mathscr{P}\int_{-1}^{1}\frac{\hat{\gamma}(\xi)\,d\xi}{x-\xi} = f_0(x)+\frac{i\omega\bar{\Gamma}}{2\pi}\{\tfrac{1}{2}(1+x)\ln(1+x)-\tfrac{1}{2}(1-x)\ln(1-x)+1\}$$
$$\equiv F_0(x). \tag{4.4}$$

Clearly $F_0(x)$ is bounded over all of $-1 \leqslant x \leqslant 1$. It may be shown (see Muskhelishvili 1953) that the general form of the solution to equation (4.4) is

$$\hat{\gamma}(x) = c_l\left(\frac{1-x}{1+x}\right)^{\frac{1}{2}}+c_t\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}}+g(x), \tag{4.5}$$

where c_l and c_t are constants, $g(x)$ is bounded, and $g(-1) = g(1) = 0$. It then follows immediately that $\gamma(x)$ must have the form

$$\gamma(x) = c_l\left(\frac{1-x}{1+x}\right)^{\frac{1}{2}}+c_t\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}}+g(x)-i\omega\bar{\Gamma}\left(\frac{1+x}{2}\right). \tag{4.6}$$

4.2. *Ordering of the integral equation for $k \gg 1$*

Let

$$\left.\begin{aligned} w_p(x) &= \frac{1}{2\pi}\mathscr{P}\int_{-1}^{1}\frac{\hat{\gamma}(\xi)\,d\xi}{x-\xi}, \\ w_r(x) &= \mathscr{P}\int_{-1}^{1}\gamma(\xi)\,\mathscr{K}(x-\xi)\,d\xi, \end{aligned}\right\} \tag{4.7}$$

where $$\mathscr{K}(x-\xi) = \frac{e^{k(x-\xi)}}{2\pi}[kK_0(\beta|x-\xi|)-\beta\,\mathrm{sgn}\,(x-\xi)\,K_1(\beta|x-\xi|)].$$

Then by equation (3.10), $\gamma(x)$ must satisfy

$$(1+i\omega x)\,\alpha_0+i\omega h_0-\frac{i\omega\bar{\Gamma}}{2\pi}\int_1^\infty \exp\left[-i\omega(\xi-1)\right]\left[1-\exp\left(-\frac{k(\xi-x)^2}{2(\xi-1)}\right)\right]\frac{d\xi}{\xi-x}$$
$$+\frac{i\omega\bar{\Gamma}}{4\pi}\mathscr{P}\int_{-1}^{1}\frac{(1+\xi)\,d\xi}{x-\xi} = w_p(x)+w_r(x). \tag{4.8}$$

To determine the orders of magnitude of the various contributions to $w_r(x)$, define

$$w_{r_l}(x) = \mathscr{P}\int_{-1}^{1}\left(\frac{1-\xi}{1+\xi}\right)^{\frac{1}{2}}\mathscr{K}(x-\xi)\,d\xi,$$

$$w_{r_t}(x) = \mathscr{P}\int_{-1}^{1}\left(\frac{1+\xi}{1-\xi}\right)^{\frac{1}{2}}\mathscr{K}(x-\xi)\,d\xi,$$

$$w_{r_g}(x) = \mathscr{P}\int_{-1}^{1}g(\xi)\,\mathscr{K}(x-\xi)\,d\xi,$$

$$w_{r_\Gamma}(x) = \mathscr{P}\int_{-1}^{1}(1+\xi)\,\mathscr{K}(x-\xi)\,d\xi,$$

so that $$w_r(x) = c_l w_{r_l}(x)+c_t w_{r_t}(x)+w_{r_g}(x)-\tfrac{1}{2}i\omega\bar{\Gamma}w_{r_\Gamma}(x). \tag{4.9}$$

The procedure for obtaining the values of w_{r_l}, w_{r_t}, etc., for $k \gg 1$, is lengthy but straightforward. For illustrative purposes, the order of magnitude of $w_{r_l}(x)$ is computed in Appendix 1. Further details of the computations are given by Crimi (1964). It is found that, for x not near the end-points ± 1,

$$w_{r_l}(x) = \frac{-1}{4(1+x)\sqrt{(2\pi k)}} \times \int_{-1}^{x} e^{-i\omega(x-\xi)} \left[\left\{ \frac{1+\xi}{(1-\xi)(x-\xi)} \right\}^{\frac{1}{2}} + 2i\omega \left\{ \frac{(1-\xi)(x-\xi)}{1+\xi} \right\}^{\frac{1}{2}} \right] d\xi + O(k^{-1}),$$

$$w_{r_t}(x) = \frac{1}{2\sqrt{(2\pi k)}} \int_{-1}^{x} \frac{e^{-i\omega(x-\xi)}\, d\xi}{(1-\xi)^{\frac{3}{2}}\{(x-\xi)(1+\xi)\}^{\frac{1}{2}}} + O(k^{-1}),$$

$$w_{r_g}(x) = \frac{1}{2\sqrt{(2\pi k)}} \int_{-1}^{x} \frac{e^{-i\omega(x-\xi)} g'(\xi)\, d\xi}{(x-\xi)^{\frac{1}{2}}} + O(k^{-1}),$$

$$w_{r_\Gamma}(x) = \frac{1}{2\sqrt{(2\pi k)}} \int_{-1}^{x} \frac{e^{-i\omega(x-\xi)}\, d\xi}{(x-\xi)^{\frac{1}{2}}} + O(k^{-1}).$$

It is further found that all four contributions to $w_r(x)$ are of order $k^{-\frac{1}{2}}$ at $x = -1$, and that w_{r_l} and w_{r_g} are of order $k^{-\frac{1}{2}} \ln k$ at $x = 1$. It remains to discuss the orders of magnitude of w_{r_Γ} and w_{r_t} in the vicinity of $x = 1$.

Now w_{r_Γ} by itself is logarithmically infinite at $x = 1$. However, if the terms in equation (4.8) which have a coefficient $i\omega\overline{\Gamma}$ are grouped together, and collectively denoted by $w_0(x)$,

$$w_0(x) \equiv \int_{-1}^{1} (1+\xi) \left[\frac{1}{2\pi(x-\xi)} + \mathscr{K}(x-\xi) \right] d\xi - \frac{1}{\pi} \int_{1}^{\infty} \exp[-i\omega(\xi-1)] \left[1 - \exp\left(-\frac{k(\xi-x)^2}{2(\xi-1)} \right) \right] \frac{d\xi}{\xi - x},$$

then $w_0(x)$ is of order one for $x < 1$, the rotational part of the integral over the wake being exponentially small in this region. Further, it may be verified that $w_0(1)$ is also of order one. For, even though each of the two integrals making up w_0 is of order $\ln k$ in the vicinity of $x = 1$, when the integrals are combined the leading terms cancel. Therefore, the terms in equation (4.8) having a coefficient $i\omega\overline{\Gamma}$ collectively are of order one over the whole interval $-1 \leqslant x \leqslant 1$.

On the other hand, the square-root singularity has a marked effect on the value of $w_{r_t}(x)$ near $x = 1$. Specifically, it is found that

$$w_{r_t}(1) = 2(k/\pi)^{\frac{1}{2}} + O(k^{-\frac{1}{2}} \ln k).$$

In summary, then, the situation is as follows. If $k \gg 1$, the rotational contributions to the integral equation, equation (4.1), are at most $O(k^{-\frac{1}{2}})$ in comparison with the potential contributions, provided x is less than one (the rotational contribution from the wake integral is exponentially small in this region). In the immediate vicinity of the trailing edge, $x = 1$, however, one of the rotational terms becomes important. While all other rotational contributions are negligible in comparison with their potential counterparts, w_{r_t} is of order $k^{\frac{1}{2}}$ with respect to the potential terms. This effect makes it possible to deduce the form of the solution for $k \to \infty$, as is shown below.

4.3. *Derivation of the Kutta condition*

If the expression for $w_r(x)$ as given by equation (4.9) is substituted in equation (4.8), the terms multiplying $i\omega\overline{\Gamma}$ are combined and the resulting expression is rearranged, it is found that $\gamma(x)$ is required to satisfy

$$c_t w_{r_t}(x) = (1+i\omega x)\,\alpha_0 + i\omega h_0 - w_p(x) - w_{r_g}(x) \\ - c_l w_{r_l}(x) + \tfrac{1}{2} i\omega \overline{\Gamma} w_0(x) \quad (-1 \leqslant x \leqslant 1). \qquad (4.10)$$

Now for x not near unity, each of the rotational contributions is of order $k^{-\frac{1}{2}}$, and so may be neglected in comparison with its counterpart in $w_p(x)$. Thus, equation (4.10) gives that

$$0 = (1+i\omega x)\,\alpha_0 + i\omega h_0 - w_p(x) + \tfrac{1}{2} i\omega\overline{\Gamma} w_0(x) + O(k^{-\frac{1}{2}}) \quad (-1 \leqslant x < 1). \qquad (4.11)$$

Clearly, then, over all of the chord except near $x = 1$, $\gamma(x)$ must approximately satisfy the integral equation for a potential flow. Therefore, the independent potential contributions to the downwash at the plate must be of order unity. It then follows directly that $c_l - c_t$, $g(x)$ and $\overline{\Gamma}$ are all at most of order unity with respect to k.

Now consider the situation at the trailing edge. If $x = 1$ is substituted in equation (4.10), then

$$c_t w_{r_t}(1) = (1+i\omega)\,\alpha_0 + i\omega h_0 - w_p(1) - w_{r_g}(1) - (c_l - c_t)\, w_{r_l}(1) \\ - c_t w_{r_l}(1) + \tfrac{1}{2} i\omega\overline{\Gamma} w_0(1)$$

or, rearranging slightly, $\gamma(x)$ must be such that

$$c_t[w_{r_t}(1) + w_{r_l}(1)] = (1+i\omega)\,\alpha_0 + i\omega h_0 - w_p(1) - w_{r_g}(1) \\ - (c_l - c_t)\, w_{r_l}(1) + \tfrac{1}{2} i\omega\overline{\Gamma} w_0(1)$$

is satisfied. All the terms on the right-hand side of the above relation are of order unity. Therefore, the left-hand side must be of order unity as well:

$$c_t[w_{r_t}(1) + w_{r_l}(1)] = O(1).$$

But

$$w_{r_t}(1) + w_{r_l}(1) = 2(k/\pi)^{\frac{1}{2}} + O(k^{-\frac{1}{2}} \ln k).$$

Thus, it must be that c_t is at most of order $k^{-\frac{1}{2}}$, and so

$$\lim_{k\to\infty} \gamma(x) = c_l[(1-x)/(1+x)]^{\frac{1}{2}} + g(x) - \tfrac{1}{2} i\omega\overline{\Gamma}(1+x),$$

where c_l, g and $\overline{\Gamma}$ are all of order one, and $g(-1) = g(1) = 0$. Then,

$$\lim_{k\to\infty} \gamma(1) = -i\omega\overline{\Gamma}.$$

From equation (3.11), it is therefore required that $\Delta p(1) = 0$, which is the Kutta condition.

5. Calculation of the first correction to the inviscid solution

The solution for inviscid flow, with the Kutta condition applied, has been obtained by a number of investigators in various forms (e.g. see von Karman &

Sears 1938). If γ_0 denotes the value of γ in the limit $k \to \infty$, then from equation (3.10) and the above analysis γ_0 must satisfy

$$\frac{1}{2\pi}\mathscr{P}\int_{-1}^{1}\frac{\gamma_0(\xi)\,d\xi}{x-\xi} = (1+i\omega x)\,\alpha_0 + i\omega h_0 - \frac{i\omega\overline{\Gamma}_0}{2\pi}\int_1^\infty \frac{e^{-i\omega(\xi-1)}\,d\xi}{\xi - x} \quad (-1 \leqslant x \leqslant 1);$$

where
$$\overline{\Gamma}_0 = \int_{-1}^{1}\gamma_0(x)\,dx$$

and
$$\gamma_0(1) = -i\omega\overline{\Gamma}_0.$$

This equation has the solution

$$\gamma_0(x) = 2\left(\frac{1-x}{1+x}\right)^{\frac{1}{2}}\left[i\omega h_0 + \{1+i\omega(1+x)\}\,\alpha_0 - \frac{i\omega\overline{\Gamma}_0}{2\pi}\int_1^\infty\left(\frac{\xi+1}{\xi-1}\right)^{\frac{1}{2}}\frac{e^{-i\omega(\xi-1)}}{\xi-x}\,d\xi\right], \tag{5.1}$$

where
$$\overline{\Gamma}_0 = \frac{2\pi\, e^{-i\omega}\,[i\omega h_0 + (1+\tfrac{1}{2}i\omega)\,\alpha_0]}{i\omega[K_0(i\omega)+K_1(i\omega)]}.$$

If L_0 and M_0 denote the lift and moment resulting from γ_0, it then follows from equations (3.12) and (3.13) that

$$\frac{L_0\,e^{-i\omega t}}{\pi\rho U^2 b} = i\omega(\alpha_0 + i\omega h_0) + 2[\alpha_0(1+\tfrac{1}{2}i\omega) + i\omega h_0]\,C(\omega),$$

$$\frac{M_0\,e^{-i\omega t}}{\pi\rho U^2 b^2} = -\tfrac{1}{2}i\omega\alpha_0 + \tfrac{1}{8}\omega^2\alpha_0 + [\alpha_0(1+\tfrac{1}{2}i\omega) + i\omega h_0]\,C(\omega),$$

where $C(\omega)$, known as the Theodorsen function, is given by

$$C(\omega) = \frac{K_1(i\omega)}{K_0(i\omega)+K_1(i\omega)}.$$

To obtain the first correction to $\gamma_0(x)$ for $k \gg 1$, assume that $\gamma(x)$ is of the form

$$\gamma(x) = \gamma_0(x) + \frac{1}{\sqrt{(\pi k)}}\gamma_1(x). \tag{5.2}$$

If this expression for γ is substituted in equation (3.10) and terms of higher order than $k^{-\frac{1}{2}}$ are neglected, it is found that $\gamma_1(x)$ must be the solution of

$$\frac{1}{2\pi}\mathscr{P}\int_{-1}^{1}\frac{\gamma_1(\xi)\,d\xi}{x-\xi} = -\sqrt{(\pi k)}\,\mathscr{P}\int_{-1}^{1}\gamma_0(\xi)\,\mathscr{K}(x-\xi)\,d\xi - \frac{i\omega\overline{\Gamma}_1}{2\pi}\int_1^\infty\frac{e^{-i\omega(\xi-1)}\,d\xi}{\xi-x}, \tag{5.3}$$

where $\gamma_0(x)$ is given by equation (5.1), and

$$\overline{\Gamma}_1 = \int_{-1}^{1}\gamma_1(x)\,dx.$$

It is also found from ordering arguments that $\gamma_1(x)$ must satisfy the Kutta condition, removing the indeterminacy in the solution of equation (5.3).

Although equation (5.3) can in theory be inverted directly, the solution so obtained involves integrals which are not readily evaluated either analytically or numerically. A more practical method of solution is adopted in what follows.

First, a more tractable form for the right-hand side of equation (5.3) is obtained by rewriting $\gamma_0(x)$:

$$\gamma_0(x) = 2\left[i\omega h_0+\alpha_0-\frac{i\omega\overline{\Gamma}_0 e^{i\omega}}{2\pi}K_0(i\omega)\right]\left(\frac{1-x}{1+x}\right)^{\frac{1}{2}}+2i\omega\alpha_0(1-x^2)^{\frac{1}{2}}$$
$$+\frac{i\omega\overline{\Gamma}_0}{2\pi}e^{i\omega}\left(\frac{1-x}{1+x}\right)^{\frac{1}{2}}\left\{K_0(i\omega)-\int_1^\infty e^{-i\omega\xi}\left(\frac{\xi+1}{\xi-1}\right)^{\frac{1}{2}}\frac{d\xi}{\xi-x}\right\}.$$

Upon manipulation of the last term in the above expression and evaluation of certain of the resulting integrals, it is found that

$$\gamma_0(x) = 2\left[i\omega h_0+\alpha_0-\frac{i\omega\overline{\Gamma}_0}{2\pi}e^{i\omega}K_0(i\omega)\right]\left(\frac{1-x}{1+x}\right)^{\frac{1}{2}}+2i\omega\alpha_0(1-x^2)^{\frac{1}{2}}$$
$$-\frac{i\omega\overline{\Gamma}_0}{2\pi}e^{i\omega(1-x)}(\pi-\cos^{-1}x)-\frac{\omega\overline{\Gamma}_0}{\pi}e^{i\omega(1-x)}(1-x^2)^{\frac{1}{2}}\int_0^\omega e^{isx}K_0(is)\,ds.$$

With the term in $\gamma_0(x)$ which is singular at the leading edge thus separated out, the approximate expressions for the rotational contributions to the downwash given in § 4.2 may be utilized. It is then found, to the order of approximation being considered here, that

$$-\sqrt{(\pi k)}\,\mathscr{P}\int_{-1}^{1}\gamma_0(\xi)\,\mathscr{K}(x-\xi)\,d\xi = \sum_{j=1}^{4}A_j\frac{dF_j(x)}{dx}, \tag{5.4}$$

where

$$A_1 = 2\left[\alpha_0+i\omega h_0-\frac{i\omega\overline{\Gamma}_0}{2\pi}e^{i\omega}K_0(i\omega)\right],$$
$$A_2 = 2i\omega\alpha_0,$$
$$A_3 = \frac{i\omega\overline{\Gamma}_0}{\pi}e^{i\omega},$$
$$A_4 = \frac{\omega\overline{\Gamma}_0}{\pi}e^{i\omega},$$

and

$$F_1(x) = -\int_0^{\frac{1}{2}\pi}\exp\left[-i\omega(1+x)\cos^2\phi\right]\{1-\tfrac{1}{2}(1+x)\sin^2\phi\}^{\frac{1}{2}}\,d\phi,$$
$$F_2(x) = -(1+x)\int_0^{\frac{1}{2}\pi}\exp\left[-i\omega(1+x)\cos^2\phi\right]\sin^2\phi\{1-\tfrac{1}{2}(1+x)\sin^2\phi\}^{\frac{1}{2}}\,d\phi,$$
$$F_3(x) = (1+x)\,e^{-i\omega x}\int_0^{\frac{1}{2}\pi}\frac{\cos^2\phi\,d\phi}{\{1-\tfrac{1}{2}(1+x)\sin^2\phi\}^{\frac{1}{2}}},$$
$$F_4(x) = (1+x)\,e^{-i\omega x}\int_0^{\frac{1}{2}\pi}\sin^2\phi\{1-\tfrac{1}{2}(1+x)\sin^2\phi\}^{\frac{1}{2}}$$
$$\times\left[\int_0^\omega\exp\{-is[1-(1+x)\sin^2\phi]\}\,K_0(is)\,ds\right]d\phi.$$

The functions F_1, F_2, F_3 and F_4 were obtained in the above form from $w_{r_l}(x)$, etc., as given in § 4.2, by an appropriate change of variable which removed the singularities in the integrands at the end-points of the interval of integration. Note that each of these functions as written above has a derivative which is logarithmically singular at $x = 1$, while the integral which their sum approximates (i.e. the left-hand side of equation (5.4)) is actually finite at $x = 1$. However, it is easily shown that the error introduced by approximating the integral with these functions over the whole chord is of higher order in $k^{-\frac{1}{2}}$.

Equation (5.3) may now be solved numerically as follows. Let $\gamma_{1j}(x)$ denote the solution of

$$\frac{1}{2\pi}\mathscr{P}\int_{-1}^{1}\frac{\gamma_{1j}(\xi)\,d\xi}{x-\xi}=\frac{dF_j(x)}{dx}-\frac{i\omega\overline{\Gamma}_{1j}}{2\pi}\int_{1}^{\infty}\frac{e^{-i\omega(\xi-1)}\,d\xi}{\xi-x}\quad (j=1,2,3,4); \tag{5.5}$$

where $$\overline{\Gamma}_{1j}=\int_{-1}^{1}\gamma_{1j}(x)\,dx.$$

Then clearly $$\gamma_1(x)=\sum_{j=1}^{4}A_j\gamma_{1j}(x). \tag{5.6}$$

Now the derivative of each of the functions $F_1(x)$ through $F_4(x)$ is well approximated by a polynomial in x summed with a logarithmic term. Specifically, if B_j is defined by

$$B_j=-\tfrac{1}{2}\lim_{x\to 1}\frac{1}{\ln(1-x)}\frac{dF_j(x)}{dx}\quad (j=1,2,3,4),$$

then the functions $G_j(x)$,

$$G_j(x)=\frac{dF_j(x)}{dx}-B_j(1+x)\ln\left(\frac{1+x}{1-x}\right)\quad (j=1,2,3,4),$$

are not singular on $[-1,1]$ and may properly be approximated by polynomials, using the method of least squares. If polynomials of degree N are used to approximate these functions, it follows that

$$\begin{aligned}\frac{dF_j(x)}{dx}&=G_j(x)+B_j(1+x)\ln\left(\frac{1+x}{1-x}\right)\\ &\approx\sum_{K=0}^{N}a_{jK}x^K+B_j(1+x)\ln\left(\frac{1+x}{1-x}\right)\quad (j=1,2,3,4);\end{aligned} \tag{5.7}$$

where the a_{jK}'s are the solution of the set of equations

$$\sum_{K=0}^{N}[1+(-1)^{n+K}]\frac{a_{jK}}{n+K+1}=\int_{-1}^{1}x^nG_j(x)\,dx\quad (n=0,1,2,\ldots,N).$$

If the expression for dF_j/dx given in equation (5.7) is then substituted into equation (5.5), the resulting integral equation may be solved using standard techniques.

The computation of the additions to unsteady lift and moment due to viscous effects was carried out with the aid of a high-speed digital computer for various values of ω between zero and one. Owing to the large number of integrations required, accuracy was limited to three significant figures. The results of the computations give the following expressions for lift L and moment M:

$$\begin{aligned}\frac{L\,e^{-i\omega t}}{\pi\rho U^2 b}&=\alpha_0\left[2(F-\tfrac{1}{2}\omega G)+\frac{1}{\sqrt{(\pi k)}}l_{\alpha_r}+i\left\{\omega+2(G+\tfrac{1}{2}\omega F)+\frac{1}{\sqrt{(\pi k)}}l_{\alpha_i}\right\}\right]\\ &\quad+i\omega h_0\left[2F+\frac{1}{\sqrt{(\pi k)}}l_{h_r}+i\left\{\omega+2G+\frac{1}{\sqrt{(\pi k)}}l_{h_i}\right\}\right],\end{aligned} \tag{5.8}$$

$$\begin{aligned}\frac{M\,e^{-i\omega t}}{\pi\rho U^2 b^2}&=\alpha_0\left[\tfrac{1}{8}\omega^2+F-\tfrac{1}{2}\omega G+\frac{1}{\sqrt{(\pi k)}}m_{\alpha_r}+i\left\{-\tfrac{1}{2}\omega+G+\tfrac{1}{2}\omega F+\frac{1}{\sqrt{(\pi k)}}m_{\alpha_i}\right\}\right]\\ &\quad+i\omega h_0\left[F+\frac{1}{\sqrt{(\pi k)}}m_{h_r}+i\left\{G+\frac{1}{\sqrt{(\pi k)}}m_{h_i}\right\}\right],\end{aligned} \tag{5.9}$$

where F and G are the real and imaginary parts, respectively, of $C(\omega)$. The quantities l_{α_r}, l_{α_i}, etc., are tabulated in Appendix 2.

6. Discussion of results

The determination of viscous effects at large Reynolds number usually initiates with an inviscid solution, which defines the boundary-layer flow. The effect of the boundary layer on the inviscid flow itself may then be calculated. This approach is only valid if the inviscid solution is indeed the proper one, in the sense that it derives from a viscous flow in the limit of vanishing viscosity. It was demonstrated above, by means of the Oseen approximation, that the potential flow about an oscillating thin airfoil obtained by applying the Kutta condition does derive, in the limit, from a viscous flow. The only assumption made in the analysis, other than those necessary to construct the mathematical model, was that the total lift on the airfoil be finite. This assumption is certainly reasonable if consideration is limited to physically realizable flows. The use of the Oseen equations to represent the viscous flow requires further discussion, however.

The introduction of the Oseen approximation may be regarded as the replacement of the local fluid velocity by the free-stream velocity in the computation of convective derivatives. Large errors are thus incurred when the local fluid velocity differs considerably either in magnitude or direction from the free-stream velocity. If the flow is not separated, the flow in the immediate vicinity of either the leading or trailing edge differs by the greatest amount from the free-stream flow. As was found in the analysis, though, these regions are very localized, and the errors introduced there are of higher order for large Reynolds number. If, on the other hand, extensive separation occurs, the entire region of separated flow must grossly violate the assumptions of the Oseen approximation. But then, of course, the Kutta condition has no significance anyway, because the associated potential flow satisfies the wrong boundary conditions. As long as separation is confined to the immediate vicinity of the trailing edge, the Oseen approximation should be adequate, even if the boundary-layer flow is turbulent. It is only necessary to regard the computed velocity as being a mean value and the viscosity as the effective eddy viscosity to retain the correspondence with the actual flow. The Kutta condition would then still be valid.

Another situation should be mentioned in which the representation used here might be inadequate to describe the flow. Specifically, there could be an excitation of a boundary-layer instability due to the oscillatory motion of the airfoil. The possibility of such an interaction appears to be remote, though, since the characteristic frequency of the boundary layer is $O(U/\delta)$, where δ is boundary-layer thickness, while the frequencies of interest at which the airfoil oscillates are much lower, being $O(U/b)$.

The quantitative significance of the computed viscous corrections to lift and moment is difficult to assess. Unsteady boundary-layer theory has not yet produced a theoretical solution which would be adequate for purposes of comparison. Unless experimentally confirmed, then, the numerical results obtained must be regarded as mainly of qualitative interest.

The variation of the viscous corrections with Reynolds number, according to the inverse square root, is as would be expected, since the boundary layer displaces the potential flow by an amount of that order. Variation with reduced frequency is seen to occur largely in the terms out of phase with pitching amplitude or plunging velocity.

This work was partially supported by the Air Force Office of Scientific Research.

Appendix 1. The ordering of $w_{r_l}(x)$ for $x^2 < 1$

It is desired to obtain the approximate value, for $k \gg 1$ and x sufficiently less than unity, of $w_{r_l}(x)$, where

$$w_{r_l}(x) = \mathscr{P}\int_{-1}^{1}\left(\frac{1-\xi}{1+\xi}\right)^{\frac{1}{2}}\mathscr{K}(x-\xi)\,d\xi.$$

The computation is broken up into three parts, by dividing up the interval of integration, as follows: let

$$\mathscr{F}_1 = \int_{-1}^{x-(M/k)}\left(\frac{1-\xi}{1+\xi}\right)^{\frac{1}{2}}\mathscr{K}(x-\xi)\,d\xi,$$

$$\mathscr{F}_2 = \mathscr{P}\int_{x-(M/k)}^{x+(M/k)}\left(\frac{1-\xi}{1+\xi}\right)^{\frac{1}{2}}\mathscr{K}(x-\xi)\,d\xi,$$

$$\mathscr{F}_3 = \int_{x+(M/k)}^{1}\left(\frac{1-\xi}{1+\xi}\right)^{\frac{1}{2}}\mathscr{K}(x-\xi)\,d\xi,$$

so that

$$w_{r_l}(x) = \mathscr{F}_1+\mathscr{F}_2+\mathscr{F}_3.$$

The parameter M is some number, say ten, such that the asymptotic forms of $K_0(z)$ and $K_1(z)$ apply for $z \geqslant M$. Its exact value is not needed, and so will not be specified.

On the interval $-1 \leqslant \xi \leqslant x-(M/k)$, the asymptotic expansions for the Bessel functions may be substituted in $\mathscr{K}(x-\xi)$, to give

$$\mathscr{K}(x-\xi) = \frac{-e^{-i\omega(x-\xi)}}{4(2\pi k)^{\frac{1}{2}}}[(x-\xi)^{-\frac{3}{2}}+2i\omega(x-\xi)^{-\frac{1}{2}}]+O(k^{-\frac{3}{2}}) \quad (-1 \leqslant \xi \leqslant x-(M/k)).$$

Thus,

$$\mathscr{F}_1 = \frac{-e^{-i\omega x}}{4(2\pi k)^{\frac{1}{2}}}\int_{-1}^{x-(M/k)}\left(\frac{1-\xi}{1+\xi}\right)^{\frac{1}{2}}e^{i\omega\xi}\{(x-\xi)^{-\frac{3}{2}}+2i\omega(x-\xi)^{-\frac{1}{2}}\}\,d\xi+O(k^{-\frac{3}{2}}).$$

Upon integration by parts, it is found that

$$\mathscr{F}_1 = \frac{-e^{-i\omega M/k}}{2(2\pi M)^{\frac{1}{2}}}\left(\frac{1-x}{1+x}\right)^{\frac{1}{2}}-\frac{1}{4(1+x)(2\pi k)^{\frac{1}{2}}}\int_{-1}^{x}e^{-i\omega(x-\xi)}\left[\left\{\frac{1+\xi}{(1-\xi)(x-\xi)}\right\}^{\frac{1}{2}}+2i\omega\left\{\frac{(1-\xi)(x-\xi)}{1+\xi}\right\}^{\frac{1}{2}}\right]d\xi+O(k^{-1}).$$

In the evaluation of $\mathscr{F}_2$, $[(1-\xi)/(1+\xi)]^{\frac{1}{2}}$ may be expanded in a Taylor series about $\xi = x$

$$\mathscr{F}_2 = \frac{1}{2\pi}\left(\frac{1-x}{1+x}\right)^{\frac{1}{2}}\mathscr{P}\int_{x-(M/k)}^{x+(M/k)}\mathscr{K}(x-\xi)\,d\xi+O(k^{-1})$$

$$= \frac{1}{2\pi}\left(\frac{1-x}{1+x}\right)^{\frac{1}{2}}(e^M-e^{-M})\,K_0\left(\frac{\beta M}{k}\right)+O(k^{-1}).$$

Now
$$\left|\frac{\beta M}{k}\right| = \left|\left(1+\frac{2i\omega}{k}\right)^{\frac{1}{2}}\right| M \geqslant M,$$

and the asymptotic form of $K_0(z)$ applies for $|z| \geqslant M$. Therefore,

$$K_0\left(\frac{\beta M}{k}\right) \approx \exp\left[-\left(1+\frac{2i\omega}{k}\right)^{\frac{1}{2}} M\right]\left\{\frac{\pi}{2M(1+2i\omega/k)^{\frac{1}{2}}}\right\}^{\frac{1}{2}}$$
$$\approx \exp\left[-(1+i\omega/k)\,M\right]\sqrt{\left(\frac{\pi}{2M}\right)}.$$

Also, e^{-M} is clearly negligible in comparison with e^{M}. It then follows that

$$\mathscr{F}_2 = \frac{e^{-i\omega M/k}}{2\sqrt{(2\pi M)}}\left(\frac{1-x}{1+x}\right)^{\frac{1}{2}} + O(k^{-1}).$$

Now $\mathscr{F}_3$ may be discarded, being exponentially small:

$$|\mathscr{F}_3| \leqslant \left(\frac{1-x-(M/k)}{1+x+(M/k)}\right)^{\frac{1}{2}}\int_{x+(M/k)}^{1} |\mathscr{K}(x-\xi)|\, d\xi$$
$$\leqslant \left(\frac{1-x}{1+x}\right)^{\frac{1}{2}}\left[\frac{1}{2\pi}\left\{e^{-M}\left|K_0\left(\frac{\beta M}{k}\right)\right| - e^{-k(1-x)}\,|K_0(\beta(1-x))|\right\}\right]$$
$$= O(e^{-2M}) + O(e^{-2k}k^{-\frac{1}{2}}).$$

Summing $\mathscr{F}_1$ and $\mathscr{F}_2$, then, the terms involving M cancel, to give

$$w_{r_l}(x) = \frac{-1}{4(1+x)\sqrt{(2\pi k)}}\int_{-1}^{x} e^{-i\omega(x-\xi)}\left[\left\{\frac{1+\xi}{(1-\xi)(x-\xi)}\right\}^{\frac{1}{2}} + 2i\omega\left\{\frac{(1-\xi)(x-\xi)}{1+\xi}\right\}^{\frac{1}{2}}\right] d\xi + O(k^{-1}).$$

Appendix 2. Corrections to unsteady lift and moment

The tabulated quantities are defined in equations (5.8) and (5.9).

ω	l_{α_r}	l_{α_i}	l_{h_r}	l_{h_i}
0	1·73	0·0	1·73	0·0
0·05	1·51	−0·0580	1·51	−0·0812
0·10	1·41	0·0758	1·41	0·0319
0·20	1·37	0·356	1·37	0·266
0·30	1·38	0·601	1·38	0·459
0·40	1·40	0·825	1·41	0·629
0·50	1·42	1·04	1·43	0·787
0·60	1·45	1·25	1·45	0·941
0·70	1·48	1·45	1·47	1·09
0·80	1·51	1·66	1·49	1·24
0·81	1·52	1·68	1·49	1·26
0·82	1·52	1·70	1·49	1·27
0·90	1·55	1·86	1·51	1·39
1·00	1·59	2·06	1·53	1·54

ω	m_{α_r}	m_{α_i}	m_{h_r}	m_{h_i}
0	0·545	0·0	0·545	0·0
0·05	0·463	−0·0796	0·462	−0·0417
0·10	0·439	−0·0913	0·438	−0·0155
0·20	0·450	−0·128	0·449	0·0193
0·30	0·481	−0·189	0·472	0·0264
0·40	0·514	−0·261	0·490	0·0208
0·50	0·548	−0·338	0·504	0·00999
0·60	0·585	−0·417	0·513	−0·00283
0·70	0·624	−0·498	0·519	−0·0164
0·80	0·667	−0·579	0·523	−0·0301
0·81	0·671	−0·588	0·524	−0·0315
0·82	0·676	−0·596	0·524	−0·0329
0·90	0·713	−0·661	0·525	−0·0438
1·00	0·762	−0·744	0·526	−0·0574

REFERENCES

CHU, W. H. 1962 An aerodynamic analysis for flutter in Oseen-type viscous flow. *J. Aero. Sci.* **29**, 781.

CRIMI, P. 1964 The effects of viscosity on the flow about an oscillating flat plate. Ph.D. Thesis, Cornell U., Ithaca, N.Y.

GRAY, A., MATHEWS, G. B. & MACROBERT, T. M. 1952 *A Treatise on Bessel Functions.* London: MacMillan and Co.

HENRY, C. J. 1962 Hydrofoil flutter phenomenon and airfoil flutter theory. *IAS Paper* no. 62-54.

LAMB, H. 1945 *Hydrodynamics*, 6th ed. New York: Dover.

MUSKHELISHVILI, N. I. 1953 *Singular Integral Equations.* Groningen: P. Noordhoff N.V.

PIERCY, N. A. V. & WINNY, H. R. 1933 The skin friction of flat plates to Oseen's approximation. *Proc. Roy. Soc.* A, **140**, 543.

TAMADA, K. & MIYAGI, T. 1962 Laminar viscous flow past a flat plate set normal to the stream, with special reference to high Reynolds numbers. *J. Phys. Soc. Japan*, **17**, 373.

VON KARMAN, T. & SEARS, W. R. 1938 Airfoil theory for non-uniform motion. *J. Aero. Sci.* **5**, 379.

WOOLSTON, D. S. & CASTILE, G. E. 1951 Some effects of variations in several parameters including fluid density on the flutter speed of light uniform cantilever wings. *N.A.C.A. Tech. Note* no. 2558.

Addendum. On the boundary conditions, by S. F. SHEN

In the foregoing paper, the solution of the Oseen equations for the problem of an oscillating flat plate, in plunging and pitching motion, in an incompressible viscous uniform stream, is obtained in the high Reynolds-number limit. The equations of motion were expressed in co-ordinates fixed to the undisturbed stream, and the boundary conditions were specified in the same manner as in the classical inviscid theory, namely, along the slit representing the *mean position* of the moving plate. An argument was given for the choice of boundary conditions. In this addendum, we wish to provide a more convincing justification for the choice. It will be shown that the solution could be interpreted better in moving co-ordinates attached to the plate, not only for the case we considered, but also in the classical inviscid theory.

Our premise is that the Oseen approximation as the linearized version of the Navier–Stokes equation for small perturbations would yield a useful description here, even though the tangential velocity perturbation at the wall clearly is not small, provided that the correct differential relation for momentum balance at

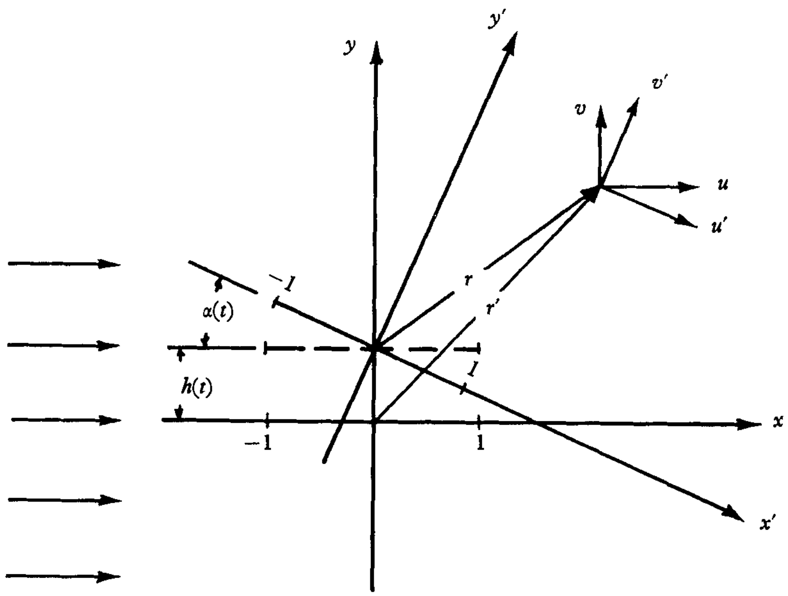

FIGURE 1 A. Fixed and moving co-ordinates.

the wall is maintained. One may wish to dispute just how 'useful' such an approximation really is, but this question has already been discussed in the foregoing paper.

On this basis, to construct the Oseen approximation for our problem, since the plate is in motion, it is most logical to start with moving co-ordinates fixed to the plate itself. On figure 1 A, we use (x, y) to denote the set of Cartesian co-ordinates fixed with respect to the free-stream of constant velocity U in the x-direction; the mean position of the plate being defined by $y = 0$, $|x| \leqslant 1$. The origin for the moving co-ordinates (x', y') is at the mid-point of the plate, so that the instantaneous position of the plate is defined by $y' = 0, |x'| \leqslant 1$. The oscillating motion of the plate includes a plunging mode $h = h_0 e^{i\omega t}$ for the mid-point, and a pitching mode $\alpha = \alpha_0 e^{i\omega t}$ around the mid-point, h_0 and α_0 being complex amplitudes and understood to be $O(\epsilon)$ $(\epsilon \ll 1)$.

The relation between the velocity components (u, v) in the 'fixed' co-ordinates and (u', v') in the 'moving' co-ordinates is then

$$\left.\begin{aligned} u - v\alpha &= u' + y'\dot{\alpha} - \dot{h}\alpha, \\ u\alpha + v &= v' - x'\dot{\alpha} + \dot{h}, \end{aligned}\right\} \qquad (1\,\mathrm{A})$$

where a 'dot' denotes the time derivative. The relation between the accelerations may also be obtained, leading to the following equations of motion in the moving co-ordinates:

$$\left.\begin{aligned} \frac{D'}{Dt}u' &= -\frac{1}{\rho}\frac{\partial}{\partial x'}p+\nu\nabla'^2 u' - [2v'\dot{\alpha}+y'\ddot{\alpha}-x'\dot{\alpha}^2-\ddot{h}\alpha], \\ \frac{D'}{Dt}v' &= -\frac{1}{\rho}\frac{\partial}{\partial y'}p+\nu\nabla'^2 v' - [-2u'\dot{\alpha}-x'\ddot{\alpha}-y'\dot{\alpha}^2+\ddot{h}], \\ \frac{\partial}{\partial x'}u'+\frac{\partial}{\partial y'}v' &= 0, \end{aligned}\right\} \tag{2 A}$$

where primed quantities refer to the moving co-ordinates, i.e.

$$\frac{D'}{Dt} = \frac{\partial}{\partial t}+u'\frac{\partial}{\partial x'}+v'\frac{\partial}{\partial y'},$$

$$\nabla'^2 = \frac{\partial^2}{\partial x'^2}+\frac{\partial^2}{\partial y'^2},$$

and $\ddot{\alpha}$ and $\ddot{h}$ are the accelerations of α and h, respectively.

The mathematical problem is to find a solution for equation (2 A) with the boundary conditions at the plate, for

$$y' = 0, \quad |x'| \leqslant 1, \quad u' = v' = 0,$$

and the condition at infinity, as

$$r' \equiv \sqrt{(x'^2+y'^2)} \to \infty, \quad u \to U, \quad v \to 0,$$

which, by equation (1 A), becomes

$$\left.\begin{aligned} U &= u'+y'\dot{\alpha} \\ U\alpha &= v'-x'\dot{\alpha}+\dot{h} \end{aligned}\right\} \quad \text{as} \quad r' \to \infty, \tag{3 A}$$

after omitting $\dot{h}\alpha$ as being $O(\epsilon^2)$.

We next restrict equation (2 A) to a finite region $r' = R$, say, where $R \sim O(1)$. Then the terms $x'\dot{\alpha}^2$ and $y'\dot{\alpha}^2$ are $O(\epsilon^2)$, as, of course, is $\ddot{h}\alpha$. At large Reynolds numbers, except in the immediate neighbourhood of the plate, all disturbances are accepted to be $O(\epsilon)$. Thus we introduce u'' and v'' in the moving co-ordinates, as small perturbations from the uniform stream

$$\left.\begin{aligned} u'' &= u'-(U-y'\dot{\alpha}), \\ v'' &= v'-(U\alpha+x'\dot{\alpha}-\dot{h}). \end{aligned}\right\} \tag{4 A}$$

Assuming $u' \sim O(U)$, $v' \sim O(U\epsilon)$, equation (2 A) may be formally linearized to retain only terms of $O(\epsilon)$

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x'}\right)\begin{pmatrix}u''\\v''\end{pmatrix} = -\frac{1}{\rho}\begin{pmatrix}\partial/\partial x'\\ \partial/\partial y'\end{pmatrix}p+\nu\nabla'^2\begin{pmatrix}u''\\v''\end{pmatrix} \tag{5 A}$$

within the region $r' = R \sim O(1)$. Hence, the linearization, in moving co-ordinates and in a finite region, leads to a set of equations similar to the equations usually obtained in fixed co-ordinates (x, y).

The boundary conditions at the plate in terms of u'' and v'' are for

$$y' = 0, \quad |x'| \leqslant 1, \quad u'' = -U, \quad v'' = \dot{h} - U\alpha - x'\dot{\alpha}. \tag{6 A}$$

As already mentioned, we shall consider equation (5 A) a useful approximation if equation (2 A) and equation (5 A) agree on the plate. The exact equation requires that, on the plate,

$$\left.\begin{aligned} \frac{D'}{Dt}u'' &= -\frac{1}{\rho}\frac{\partial p}{\partial x'} + \nu\frac{\partial^2 u''}{\partial y'^2} + O(\epsilon^2), \\ \frac{D'}{Dt}v'' &= -\frac{1}{\rho}\frac{\partial p}{\partial y'} + \nu\frac{\partial^2 v''}{\partial y'^2} - U\dot{\alpha} + O(\epsilon^2), \end{aligned}\right\} \tag{7 A}$$

after the right-hand sides are evaluated with equation (6 A). But on the plate, again from equation (6 A), the left-hand sides are

$$\frac{D'}{Dt}u'' = \frac{\partial u''}{\partial t} = 0,$$

$$\frac{D'}{Dt}v'' = \ddot{h} - U\dot{\alpha} - x'\ddot{\alpha} - u'\dot{\alpha} = \ddot{h} - U\dot{\alpha} - x'\ddot{\alpha} = \frac{\partial v''}{\partial t}.$$

Hence equation (7 A) is equivalent to

$$-\frac{1}{\rho}\frac{\partial p}{\partial x'} + \nu\frac{\partial^2 u''}{\partial y'^2} = 0,$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial y'} + \nu\frac{\partial^2 v''}{\partial y'^2} = \frac{\partial v''}{\partial t},$$

which are in fact satisfied by similarly applying equation (5 A) to the plate. Hence, we conclude that equation (5 A) is the desired Oseen approximation in moving co-ordinates, in a finite region $r' = R \sim O(1)$.

For the region beyond $r' = R$, let us revert to fixed co-ordinates (x, y), and introduce perturbations from the uniform flow

$$\tilde{u} = U - u, \quad \tilde{v} = v.$$

Since, for $r > R$, $\tilde{u}, \tilde{v} \sim O(\epsilon)$ at most, surely the usual linearization is justified

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\begin{pmatrix}\tilde{u}\\ \tilde{v}\end{pmatrix} = -\frac{1}{\rho}\begin{pmatrix}\partial/\partial x\\ \partial/\partial y\end{pmatrix} p + \nu\nabla^2\begin{pmatrix}\tilde{u}\\ \tilde{v}\end{pmatrix}. \tag{8 A}$$

The equations of continuity in both the moving and fixed co-ordinates are, of course, also identical in appearance.

Consequently, we now propose to describe the flow by two sets of equations, one for the near region using moving co-ordinates, and one for the far region using fixed co-ordinates. The complete solution requires:

(i) a solution (u'', v'') of equation (5 A) satisfying equation (6 A) at the plate and decaying to $O(\epsilon)$ as $r' = R \sim O(1)$,

(ii) a solution $(\tilde{u}, \tilde{v})$ of equation (8 A) matching the above (after being properly transformed by equation (1 A)) along the contour $r' = R$, and tending to zero as $r \to \infty$.

The observation may be made that, since (u'', v'') and $(\tilde{u}, \tilde{v})$ satisfy the same differential equations (although in different co-ordinates), it is sufficient to construct only a single solution (u'', v'') of equation (5 A) satisfying equation (6 A) *and tending to zero as* $r' \to \infty$. Such a solution certainly meets the requirements of (i). In addition, along the matching contour $r' = R$ it could be used directly as the values of $(\tilde{u}, \tilde{v})$ to provide the boundary condition of (ii), with the error $O(\epsilon^2)$, which error is partly due to the transformation equation (1 A) and partly due to the distance, $O(\epsilon)$, between the points (x', y') and (x, y) along the matching contour. It follows that to an accuracy of $O(\epsilon)$, the very same solution (u'', v'') in (x', y') co-ordinates actually may be used as that for $(\tilde{u}, \tilde{v})$ in (x, y) co-ordinates in the far region.

In other words, the problem solved in the foregoing paper might be regarded as a composite solution, representing the near field in moving co-ordinates and the far field in fixed co-ordinates. We may note, however, that, in the pure plunging oscillation, the entire solution could be interpreted in moving co-ordinates. The presence of the pitching motion α is the only reason that a matching along $r' = R$ has to be introduced, in order to render the perturbation equation in the form of equation (5 A).

In the classical inviscid theory written in fixed co-ordinates, the usual justification for applying the boundary condition at the mean position of the plate rests on the fact that the normal velocity component alone determines the solution. Since the normal velocity is continuous across the plate, the error in neglecting the plate displacement is $O(\epsilon^2)$. However, this argument could not be carried over to the case where the tangential velocity component, discontinuous across the plate, is also needed for the solution, as in the presence of a boundary layer. Therefore, to achieve a unified point of view it seems preferable to interpret the inviscid theory also in terms of moving co-ordinates attached to the plate. The calculation of the pressure distribution along the plate, etc., naturally remains unchanged.